

# ALGEBRA IN THE SUPEREXTENSIONS OF SEMILATTICES

TARAS BANAKH AND VOLODYMYR GAVRYLKIV

**ABSTRACT.** Given a semilattice  $X$  we study the algebraic properties of the semigroup  $v(X)$  of upfamilies on  $X$ . The semigroup  $v(X)$  contains the Stone-Čech extension  $\beta(X)$ , the superextension  $\lambda(X)$ , and the space of filters  $\varphi(X)$  on  $X$  as closed subsemigroups. We prove that  $v(X)$  is a semilattice iff  $\lambda(X)$  is a semilattice iff  $\varphi(X)$  is a semilattice iff the semilattice  $X$  is finite and linearly ordered. We prove that the semigroup  $\beta(X)$  is a band if and only if  $X$  has no infinite antichains, and the semigroup  $\lambda(X)$  is commutative if and only if  $X$  is a bush with finite branches.

## INTRODUCTION

One of powerful tools in the modern Combinatorics of Numbers is the method of ultrafilters based on the fact that each (associative) binary operation  $*$  :  $X \times X \rightarrow X$  defined on a discrete topological space  $X$  extends to a right-topological (associative) operation  $*$  :  $\beta(X) \times \beta(X) \rightarrow \beta(X)$  on the Stone-Čech compactification  $\beta(X)$  of  $X$ , see [9], [11]. The Stone-Čech extension  $\beta(X)$  is the space of ultrafilters on  $X$ . The extension of the operation from  $X$  to  $\beta(X)$  can be defined by the simple formula:

$$(1) \quad \mathcal{U} * \mathcal{V} = \left\langle \bigcup_{x \in U} x * V_x : U \in \mathcal{U}, (V_x)_{x \in U} \in \mathcal{V}^U \right\rangle,$$

where  $\langle \mathcal{B} \rangle = \{A \subset X : \exists B \in \mathcal{B} \ B \subset A\}$  is the upper closure of a family  $\mathcal{B}$ . In this case  $\mathcal{B}$  is called a *base* of  $\langle \mathcal{B} \rangle$ .

Endowed with the so-extended operation, the Stone-Čech compactification  $\beta(X)$  becomes a compact right-topological semigroup. The algebraic properties of this semigroup (for example, the existence of idempotents or minimal left ideals) have important consequences in combinatorics of numbers, see [9], [11].

In [8] it was observed that the binary operation  $*$  extends not only to  $\beta(X)$  but also to the space  $v(X)$  of all upfamilies on  $X$ . By definition, a family  $\mathcal{F}$  of non-empty subsets of a discrete space  $X$  is called an *upfamily* if for any sets  $A \subset B \subset X$  the inclusion  $A \in \mathcal{F}$  implies  $B \in \mathcal{F}$ . The space  $v(X)$  is a closed subspace of the double power-set  $\mathcal{P}(\mathcal{P}(X))$  endowed with the compact Hausdorff topology of the Tychonoff power  $\{0, 1\}^{\mathcal{P}(X)}$ . In the papers [7], [8], [1]–[4] the space  $v(X)$  was denoted by  $G(X)$  and its elements were called inclusion hyperspaces<sup>1</sup>. The extension of a binary operation  $*$  from  $X$  to  $v(X)$  can be defined in the same way as for ultrafilters, i.e., by the formula (1) applied to any two upfamilies  $\mathcal{U}, \mathcal{V} \in v(X)$ . If  $X$  is a semigroup, then  $v(X)$  is a compact Hausdorff right-topological semigroup containing  $\beta(X)$  as closed subsemigroups. The algebraic properties of this semigroups were studied in details in [8].

The space  $v(X)$  of upfamilies over a discrete space  $X$  contains many interesting subspaces. First we recall some definitions. An upfamily  $\mathcal{A} \in v(X)$  is defined to be

- a *filter* if  $A_1 \cap A_2 \in \mathcal{A}$  for all sets  $A_1, A_2 \in \mathcal{A}$ ;
- an *ultrafilter* if  $\mathcal{A} = \mathcal{A}'$  for any filter  $\mathcal{A}' \in v(X)$  containing  $\mathcal{A}$ ;
- *linked* if  $A \cap B \neq \emptyset$  for any sets  $A, B \in \mathcal{A}$ ;
- *maximal linked* if  $\mathcal{A} = \mathcal{A}'$  for any linked upfamily  $\mathcal{A}' \in v(X)$  containing  $\mathcal{A}$ .

1991 *Mathematics Subject Classification.* 06A12, 20M10.

*Key words and phrases.* semilattice, band, commutative semigroup, the space of upfamilies, the space of filters, the space of maximal linked systems, superextension.

<sup>1</sup>We decided to change the terminology and notation after discovering the paper [12, 2.7.4] that discusses monadic properties of the up-set functor  $v$ .

By  $\varphi(X)$ ,  $\beta(X)$ ,  $N_2(X)$ , and  $\lambda(X)$  we denote the subspaces of  $v(X)$  consisting of filter, ultrafilters, linked upfamilies, and maximal linked upfamilies, respectively. The space  $\lambda(X)$  is called *the superextension* of  $X$ , see [10], [14]. In [8] it was observed that for a discrete semigroup  $X$  the subspaces  $\varphi(X)$ ,  $\beta(X)$ ,  $N_2(X)$ ,  $\lambda(X)$  are closed subsemigroups of the semigroup  $v(X)$ . The following diagram describes the inclusion relations between these subspaces of  $v(X)$  (an arrow  $A \rightarrow B$  indicates that  $A$  is a subset of  $B$ ).

$$\begin{array}{ccccc} \beta(X) & \longrightarrow & \lambda(X) & & \\ \downarrow & & \downarrow & & \\ \varphi(X) & \longrightarrow & N_2(X) & \longrightarrow & v(X) \end{array}$$

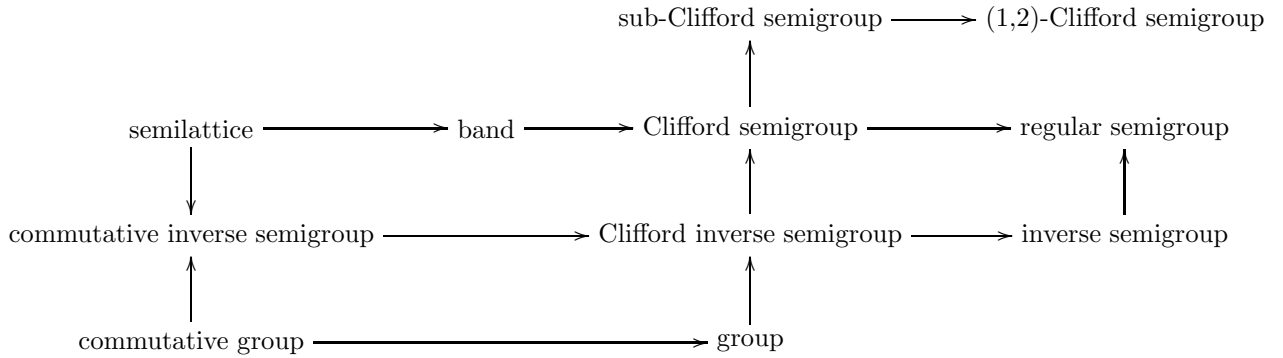
In [8], [1] — [4] we studied the properties of the compact right-topological semigroup  $v(X)$  and its subsemigroups for groups  $X$ . In this paper we shall study the algebraic structure of the semigroups  $\lambda(X)$ ,  $\varphi(X)$ ,  $N_2(X)$ , and  $v(X)$  for semilattices  $X$ .

Let us recall that a *semilattice* is a commutative idempotent semigroup. Idempotent semigroups are called *bands*. So, in a band each element  $x$  is an *idempotent*, which means that  $xx = x$ . A semigroup  $S$  is *linear* if  $xy \in \{x, y\}$  for any elements  $x, y \in X$ . It follows that each linear semigroup  $S$  is a band. Each (linear) semilattice is partially (linearly) ordered by the relation  $\leq$  defined by  $x \leq y$  iff  $xy = x$ .

A semigroup  $S$  is *cancellative* if for each element  $a \in S$  the left shift  $l_a : S \rightarrow S$ ,  $l_a : x \mapsto ax$ , and the right shift  $r_a : S \rightarrow S$ ,  $r_a : x \mapsto xa$ , are injective. A semigroup  $S$  is called *Clifford* (resp. *sub-Clifford*) if  $S$  is a union of groups (resp. of cancellative semigroups). Observe that a subsemigroup of a sub-Clifford semigroup is sub-Clifford and a finite semigroup  $S$  is Clifford if and only if it is sub-Clifford. It is easy to see that a semigroup  $S$  is sub-Clifford if and only if for every natural numbers  $n, m$  it is  $(n, m)$ -Clifford in the sense that for any element  $x \in S$  the equality  $x^{n+1} = x^{m+1}$  implies  $x^n = x^m$ .

A semigroup  $S$  is called a *regular semigroup* if  $a \in aSa$  for any  $a \in S$ . Such a semigroup  $S$  is called an *inverse semigroup* if  $ab = ba$  for any idempotents  $a, b \in S$ . Observe that each band is a Clifford semigroup and every Clifford semigroup is sub-Clifford and regular. An inverse semigroup with a unique idempotent is a group.

These algebraic properties relate as follows:



In this paper we shall characterize semigroups  $X$  whose extensions  $v(X)$ ,  $\lambda(X)$ ,  $\varphi(X)$  or  $N_2(X)$  are bands, linear semigroups, commutative semigroups, or semilattices. In Section 5 we shall characterize lattices  $X$  whose extensions  $v(X)$ ,  $\lambda(X)$ ,  $\varphi(X)$  are lattices. The results obtained in this paper will be applied in the paper [5] devoted to the superextensions of inverse semigroups.

## 1. SEMIGROUPS WHOSE EXTENSIONS ARE BANDS

In this section we shall characterize semigroups  $X$  whose extensions  $v(X)$ ,  $\lambda(X)$  or  $\varphi(X)$  are bands. Let us recall that a semigroup  $S$  is a (linear) band if  $xx = x$  for all  $x \in X$  (and  $xy \in \{x, y\}$  for all  $x, y \in X$ ).

Let us recall that an element  $a$  of a semigroup  $S$  is *regular* in  $S$  if  $a \in aSa$ . It is clear that each idempotent is a regular element.

**Theorem 1.1.** *For a semigroup  $X$  the following conditions are equivalent:*

- (1)  $X$  is linear;
- (2)  $v(X)$  is a band;
- (3)  $\varphi(X)$  is a band;
- (4)  $\lambda(X)$  is a band.

*Proof.* (1)  $\Rightarrow$  (2) Assume that the semigroup  $X$  is linear. To show that  $v(X)$  is a band, we should check that  $\mathcal{A} * \mathcal{A} = \mathcal{A}$  for any upfamily  $\mathcal{A} \in v(X)$ . Since  $X$  is linear, for any  $A \in \mathcal{A}$  we get  $A = A * A \in \mathcal{A} * \mathcal{A}$  and hence  $\mathcal{A} \subset \mathcal{A} * \mathcal{A}$ .

To show that  $\mathcal{A} \supset \mathcal{A} * \mathcal{A}$ , fix any basic subset  $B = \bigcup_{x \in A} x * A_x \in \mathcal{A} * \mathcal{A}$  where  $A \in \mathcal{A}$  and  $A_x \in \mathcal{A}$  for all  $x \in A$ .

Now we consider two cases.

- (i) There is  $x \in A$  such that  $xa = a$  for all  $a \in A_x$ . In this case  $\mathcal{A} \ni A_x = x * A_x \subset B$  and thus  $B \in \mathcal{A}$ .
- (ii) For every  $x \in A$  there is  $a \in A_x$  such that  $xa \neq a$  and hence  $xa = x$  (as  $X$  is linear). In this case  $\mathcal{A} \ni A \subset \bigcup_{x \in A} x * A_x = B$  and hence  $B \in \mathcal{A}$ .

The implications (2)  $\Rightarrow$  (3, 4) are trivial.

(3)  $\Rightarrow$  (1) Assume that  $\varphi(X)$  is a band. Then  $X$ , being a subsemigroup of  $\varphi(X)$ , also is a band. To show that  $X$  is linear, take any two points  $x, y \in X$  and consider the filter  $\mathcal{F} = \langle \{x, y\} \rangle \in \varphi(X)$ . Being an idempotent, the filter  $\mathcal{F}$  is regular in  $v(X)$ . Consequently, we can find an upfamily  $\mathcal{A} \in v(X)$  such that  $\mathcal{F} * \mathcal{A} * \mathcal{F} = \mathcal{F}$ . It follows that there are sets  $A_x, A_y \in \mathcal{A}$  such that  $(xA_x \cup yA_y) \cdot \{x, y\} \subset \{x, y\}$ . In particular, for every  $a_x \in A_x$  we get  $xa_xy \in \{x, y\}$ . If  $xa_xy = x$ , then  $xy = xa_xyy = xa_xy = x$ . If  $xa_xy = y$ , then  $xy = xxa_xy = xa_xy = y$ , witnessing that the band  $X$  is linear.

(4)  $\Rightarrow$  (1) Assume that  $\lambda(X)$  is a band. Then  $X$ , being a subsemigroup of  $\lambda(X)$ , is a band as well. Assuming that the band  $X$  is not linear, we can find two points  $x, y \in X$  such that  $xy \notin \{x, y\}$ . We claim that the maximal linked system  $\mathcal{L} = \langle \{x, y\}, \{x, xy\}, \{y, xy\} \rangle \in \lambda(X)$  is not an idempotent. We shall prove more: the element  $\mathcal{L}$  is not regular in the semigroup  $v(X)$ . Assuming the converse, we can find an upfamily  $\mathcal{A} \in v(X)$  such that  $\mathcal{L} * \mathcal{A} * \mathcal{L} = \mathcal{L}$ . It follows from  $\{x, y\} \in \mathcal{L} = \mathcal{L} * \mathcal{A} * \mathcal{L}$  that  $\{x, y\} \supset \bigcup_{u \in L} u * B_u$  for some set  $L \in \mathcal{L}$  and some sets  $B_u \in \mathcal{A} * \mathcal{L}$ ,  $u \in L$ . The linked property of family  $\mathcal{L}$  implies that the intersection  $L \cap \{x, xy\}$  contains some point  $u$ . Now for the set  $B_u \in \mathcal{A} * \mathcal{L}$  find a set  $A \in \mathcal{A}$  and a family  $(L_a)_{a \in A} \in \mathcal{L}^A$  such that  $B_u \supset \bigcup_{a \in A} a * L_a$ . Fix any point  $a \in A$  and a point  $v \in L_a \cap \{y, xy\}$ . Then  $uav \in uaL_a \subset uB_u \subset \{x, y\}$ . Since  $u \in \{x, xy\}$  and  $v \in \{y, xy\}$ , the element  $uav$  is equal to  $xbv$  for some element  $b \in \{a, ya, ax, yax\}$ . So,  $xbv \in \{x, y\}$ . If  $xbv = x$ , then  $xy = xbyy = xby = x \in \{x, y\}$ . If  $xbv = y$ , then  $xy = xby = xby = y \in \{x, y\}$ . In both cases we obtain a contradiction with the choice of the points  $x, y \notin \{x, y\}$ .  $\square$

Observe that the proof of Theorem 1.1 yields a bit more, namely:

**Proposition 1.2.** *For a band  $X$  the following conditions are equivalent:*

- (1)  $X$  is linear;
- (2) each element of  $\varphi(X)$  is regular in  $v(X)$ ;
- (3) each element of  $\lambda(X)$  is regular in  $v(X)$ .

The linearity of a semilattice  $X$  can be also characterized via the (1, 2)-Clifford property of the semigroups  $\varphi(X)$  and  $\lambda(X)$ .

**Theorem 1.3.** *For a semilattice  $X$  the following conditions are equivalent:*

- (1)  $X$  is linear;
- (2)  $\varphi(X)$  is (1, 2)-Clifford;
- (3)  $\lambda(X)$  is (1, 2)-Clifford.

*Proof.* The implications  $(1) \Rightarrow (2, 3)$  follow from Theorem 1.1 because each band is a  $(1, 2)$ -Clifford semigroup.

$(2, 3) \Rightarrow (1)$  Assume that the semilattice  $X$  is not linear. Then  $X$  contains two elements  $x, y \in X$  such that  $yx = xy \notin \{x, y\}$ .

Consider the filter  $\mathcal{F} = \langle \{x, y\} \rangle$  and observe that  $\mathcal{F} \neq \mathcal{F} \cdot \mathcal{F} = \langle \{x, xy, y\} \rangle = \mathcal{F} \cdot \mathcal{F} \cdot \mathcal{F}$ , which means that the semigroup  $\varphi(X)$  is not  $(1, 2)$ -Clifford.

To see that  $\lambda(X)$  is not  $(1, 2)$ -Clifford, consider the maximal linked system  $\mathcal{L} = \langle \{x, y\}, \{x, xy\}, \{y, xy\} \rangle \in \lambda(X)$  and observe that  $\mathcal{L} \neq \mathcal{L} \cdot \mathcal{L} = \langle \{xy\} \rangle = \mathcal{L} \cdot \mathcal{L} \cdot \mathcal{L}$ .  $\square$

Next we characterize semigroups  $X$  whose Stone-Ćech extension  $\beta(X)$  is a band. A sequence  $(x_n)_{n \in \omega}$  of points of some set  $X$  is called *injective* if  $x_n \neq x_m$  for any distinct numbers  $n, m \in \omega$ .

**Theorem 1.4.** *For a band  $X$  the semigroup  $\beta(X)$  is a band if and only if for each injective sequence  $(x_n)_{n \in \omega}$  in  $X$  there are numbers  $n < m$  such that  $x_n x_m \in \{x_n, x_m\}$ .*

*Proof.* To prove the “only if” part, assume that  $(x_n)_{n \in \omega}$  is an injective sequence in  $X$  such that  $x_n x_m \notin \{x_n, x_m\}$  for all  $n < m$ . We claim that there is an infinite subset  $\Omega \subset \omega$  such that  $x_n x_m \neq x_k$  for any numbers  $n, m, k \in \Omega$  with  $n < m$ . For this we shall apply the famous Ramsey Theorem. Consider the 4-coloring  $\chi : [\omega]^3 \rightarrow 4 = \{0, 1, 2, 3\}$  of the set  $[\omega]^3 = \{(k, n, m) \in \omega^3 : k < n < m\}$ , defined by

$$\chi(k, n, m) = \begin{cases} 1 & \text{if } x_k x_n = x_m, \\ 2 & \text{if } x_k x_m = x_n, \\ 3 & \text{if } x_n x_m = x_k, \\ 0 & \text{otherwise.} \end{cases}$$

By the Ramsey Theorem [11, 5.1], there is an infinite set  $\Omega \subset \omega$  such that  $\chi(\Omega^3 \cap [\omega]^3)$  is a singleton. It follows from the definition of the coloring  $\chi$  that this singleton is  $\{0\}$ , which means that for any numbers  $k, n, m \in \Omega$  with  $n < m$  and  $k \notin \{n, m\}$  we get  $x_n x_m \neq x_k$ . Since  $x_n x_m \notin \{x_n, x_m\}$  for any numbers  $n < m$ , we conclude that  $x_n x_m \neq x_k$  for any numbers  $k, n, m \in \Omega$  with  $n < m$ .

Now take any free ultrafilter  $\mathcal{A}$  that contains the set  $A = \{x_n\}_{n \in \Omega}$ . Then for every  $n \in \omega$  the set  $A_{>n} = \{x_m : n < m \in \Omega\}$  belongs to the ultrafilter  $\mathcal{A}$ . The choice of the sequence  $A = \{x_n\}_{n \in \Omega}$  guarantees that  $A \cap \bigcup_{n \in \Omega} x_n * A_{>n} = \emptyset$ , which implies that  $\mathcal{A} \neq \mathcal{A} * \mathcal{A}$  and hence the ultrafilter  $\mathcal{A}$  is not an idempotent in  $\beta(X)$ .

To prove the “if” part, assume that  $\beta(X)$  is not a band and find an ultrafilter  $\mathcal{F} \in \beta(X)$  with  $\mathcal{F} * \mathcal{F} \neq \mathcal{F}$ . In particular,  $\mathcal{F} * \mathcal{F} \not\subseteq \mathcal{F}$ . This implies that for some  $A \in \mathcal{F}$  and  $\{A_x\}_{x \in A} \subset \mathcal{F}$  the set  $\bigcup_{x \in A} x * A_x \notin \mathcal{F}$ .

Consider the set  $X_{\mathcal{F}}^{\uparrow} = \{x \in X : \uparrow x \in \mathcal{F}\}$  where  $\uparrow x = \{y \in X : xy = x\}$ . We claim that  $X_{\mathcal{F}}^{\uparrow} \notin \mathcal{F}$ . Assuming that  $X_{\mathcal{F}}^{\uparrow} \in \mathcal{F}$ , we conclude that  $A \cap X_{\mathcal{F}}^{\uparrow} \in \mathcal{F}$ . This implies that  $\uparrow a \in \mathcal{F}$  and  $\uparrow a \cap A_a \in \mathcal{F}$  for any  $a \in A \cap X_{\mathcal{F}}^{\uparrow}$ . Therefore  $a * (\uparrow a \cap A_a) = \{a\}$  and hence

$$\bigcup_{x \in A} x * A_x \supset \bigcup_{x \in A \cap X_{\mathcal{F}}^{\uparrow}} x * (\uparrow x \cap A_x) = \bigcup_{x \in A \cap X_{\mathcal{F}}^{\uparrow}} \{x\} = A \cap X_{\mathcal{F}}^{\uparrow} \in \mathcal{F}.$$

Thus  $\bigcup_{x \in A} x * A_x \in \mathcal{F}$ . This contradiction shows that  $X_{\mathcal{F}}^{\uparrow} \notin \mathcal{F}$ .

Next, consider the set  $X_{\mathcal{F}}^{\downarrow} = \{x \in X : \downarrow x \in \mathcal{F}\}$  where  $\downarrow x = \{y \in X : xy = y\}$ . We claim that  $X_{\mathcal{F}}^{\downarrow} \notin \mathcal{F}$ . Assume that  $X_{\mathcal{F}}^{\downarrow} \in \mathcal{F}$ . Then  $A \cap X_{\mathcal{F}}^{\downarrow} \in \mathcal{F}$ . This implies that  $\downarrow a \in \mathcal{F}$  and  $\downarrow a \cap A_a \in \mathcal{F}$  for any  $a \in A \cap X_{\mathcal{F}}^{\downarrow}$ . Therefore

$$\downarrow a \cap A_a \subset a * (\downarrow a \cap A_a) \subset a * A_a \subset \bigcup_{x \in A} x * A_x.$$

Thus  $\bigcup_{x \in A} x * A_x \in \mathcal{F}$ . This contradiction shows that  $X_{\mathcal{F}}^{\downarrow} \notin \mathcal{F}$ .

Since  $\mathcal{F}$  is an ultrafilter,  $X_{\mathcal{F}}^{\uparrow} \cup X_{\mathcal{F}}^{\downarrow} \notin \mathcal{F}$  and  $Z_{\mathcal{F}} = X \setminus (X_{\mathcal{F}}^{\uparrow} \cup X_{\mathcal{F}}^{\downarrow}) \in \mathcal{F}$ . Let  $x_0 \in Z_{\mathcal{F}}$  be arbitrary and by induction, for every  $n \in \omega$  choose a point  $x_{n+1} \in Z_{\mathcal{F}} \setminus \bigcup_{i \leq n} (\uparrow x_i \cup \downarrow x_i) \in \mathcal{F}$ . Then the injective sequence  $(x_n)_{n \in \omega}$  has the required property:  $x_n x_m \notin \{x_n, x_m\}$  for  $n < m$  (which follows from  $x_m \notin \downarrow x_n \cup \uparrow x_n$ ).  $\square$

A subset  $A$  of a semigroup  $X$  is called an *antichain* if  $ab \notin \{a, b\}$  for any distinct points  $a, b \in A$ . Theorem implies the following characterization:

**Corollary 1.5.** *For a semilattice  $X$  the semigroup  $\beta(X)$  is a band if and only if each antichain in  $X$  is finite.*

## 2. SEMILATTICES WHOSE EXTENSIONS ARE COMMUTATIVE

In this section we recognize the structure of semilattices  $X$  whose extensions  $v(X)$ ,  $N_2(X)$  or  $\lambda(X)$  are commutative.

Commutative semigroups of ultrafilters were characterized in [9, 4.27] as follows:

**Theorem 2.1.** *The Stone-Ćech extension  $\beta(X)$  of a semigroup  $S$  is not commutative if and only if there are sequences  $(x_n)_{n \in \omega}$  and  $(y_n)_{n \in \omega}$  in  $X$  such that  $\{x_k y_n : k < n\} \cap \{y_k x_n : k < n\} = \emptyset$ .*

This characterization implies the following (well-known) fact:

**Corollary 2.2.** *If the Stone-Ćech extension  $\beta(X)$  of a semilattice  $X$  is commutative, then each linear subsemigroup in  $X$  is finite.*

*Proof.* Assume conversely that  $X$  contains an infinite linear subsemilattice  $L$ . Being linear,  $L$  is linearly ordered by the order  $\leq$  defined by  $x \leq y$  iff  $xy = x$ . Since  $L$  is infinite, we can apply Ramsey Theorem in order to find an injective sequence  $(z_n)_{n \in \omega}$  in  $L$ , which is either strictly increasing or strictly decreasing. Put  $x_n = z_{2n}$  and  $y_n = z_{2n+1}$  for  $n \in \omega$ . Applying Theorem 2.1 to the sequences  $(x_n)_{n \in \omega}$  and  $(y_n)_{n \in \omega}$  we conclude that the semigroup  $\beta(L)$  is not commutative. Then  $\beta(X)$  is not commutative neither.  $\square$

In spite of Theorem 2.1 the following problem seems to be open.

**Problem 2.3.** *Describe the structure of a semilattice  $X$  whose Stone-Ćech extension  $\beta(X)$  is commutative.*

A similar problem on commutativity of semigroups  $v(X)$  also is open:

**Problem 2.4.** *Characterize semigroups  $X$  whose extension  $v(X)$  is commutative.*

(It can be shown that if  $v(X)$  is commutative, then  $X$  is a commutative semigroup with finite linear idempotent band  $E = \{x \in X : xx = x\}$  and  $x^3 = x^4$  for all  $x \in X$ ).

We shall resolve this problem for bands. First we prove a useful result on multiplication of upfamilies on linear semigroups.

For a semigroup  $X$  denote by  $v^{\bullet}(X)$  the subsemigroup of  $v(X)$  consisting of all upfamilies  $\mathcal{A} \in v(X)$  such that for each set  $A \in \mathcal{A}$  there is a finite subset  $F \in \mathcal{A}$  with  $F \subset A$ .

For a semigroup  $X$  and two upfamilies  $\mathcal{A}, \mathcal{B} \in v(X)$  let

$$\mathcal{A} \otimes \mathcal{B} = \langle A * B : A \in \mathcal{A}, B \in \mathcal{B} \rangle.$$

It is clear that  $\mathcal{A} \otimes \mathcal{B} \subset \mathcal{A} * \mathcal{B}$ . In the following theorem we show that for finite linear semigroups the converse inclusion also holds.

**Theorem 2.5.** *If  $X$  is a linear semigroup, then  $\mathcal{A} * \mathcal{B} = \mathcal{A} \otimes \mathcal{B}$  for any upfamilies  $\mathcal{A} \in v^{\bullet}(X)$  and  $\mathcal{B} \in v(X)$ .*

*Proof.* On the semigroup  $X$  consider the relation  $\leq$  defined by:  $x \leq y$  iff  $yx = x$ . This relation is reflexive and transitive. For a subsets  $A \subset X$  and a point  $x \in X$  we write  $A \leq x$  if  $a \leq x$  for all  $a \in A$ . It follows from the definition of the semigroup operation  $*$  on  $v(X)$  that  $\mathcal{A} \otimes \mathcal{B} \subset \mathcal{A} * \mathcal{B}$ . To prove the reverse inclusion, fix any

basic set  $C = \bigcup_{a \in A} a * B_a \in \mathcal{A} * \mathcal{B}$  where  $A \in \mathcal{A}$  and  $B_a \in \mathcal{B}$  for all  $a \in A$ . Since  $\mathcal{A} \in v^\bullet(X)$ , we can assume that the set  $A$  is finite and hence can be enumerated as  $A = \{a_1, \dots, a_n\}$  so that  $a_i \leq a_{i+1}$  for all  $i < n$ . Now let us consider two cases.

1. For some  $i \leq n$  we get  $B_{a_i} \leq a_i$ , which means that  $a_i b = b$  for all  $b \in B_{a_i}$  and hence  $a_i * B_{a_i} = B_{a_i}$ . For every  $j \geq i$  the inequality  $B_{a_i} \leq a_i \leq a_j$  implies  $a_j * B_{a_i} = B_{a_i}$ . Consequently,  $A * B_{a_i} \subset \{a_1, \dots, a_{i-1}\} \cup B_{a_i}$ .

We can assume that  $i$  is the smallest number such that  $B_{a_i} \leq a_i$ . In this case the minimality of  $i$  implies that  $B_{a_j} \not\leq a_j$  for all  $j < i$ . This means  $b_j \not\leq a_j$  for some  $b_j \in B_{a_j}$  and hence  $a_j b_j = a_j$  (as  $a_j b_j \in \{a_j, b_j\}$  and  $a_j b_j \neq b_j$ ). Then  $a_j * B_{a_j} \ni a_j b_j = a_j$  and thus  $A * B_{a_i} \subset \{a_1, \dots, a_{i-1}\} \cup B_{a_i} \subset \bigcup_{j=1}^n a_j B_{a_j}$ , which implies that  $C \in \mathcal{A} \otimes \mathcal{B}$ .

2.  $B_{a_i} \not\leq a_i$  for all  $i \leq n$ . In this case  $a_i \in a_i * B_{a_i}$  for all  $i$ . Observe that for any  $b \in B_{a_n}$  and  $i \leq n$  we get  $a_i b \in \{a_i, b\}$  by the linearity of  $X$ . If  $a_i b \neq a_i$ , then  $a_i b = b$  and  $a_i b = b = a_n a_i b = a_n b \in a_n B_{a_n}$ . So,

$$\mathcal{A} \otimes \mathcal{B} \ni A * B_{a_n} \subset \{a_1, \dots, a_n\} \cup a_n B_{a_n} \subset \bigcup_{i=1}^n a_i B_{a_i} = C$$

and hence  $C \in \mathcal{A} \otimes \mathcal{B}$ . □

Now we are able to characterize bands  $X$  with commutative extensions  $v(X)$  and  $N_2(X)$ .

**Theorem 2.6.** *For a band  $X$  the following conditions are equivalent:*

- (1)  $X$  is a finite linear semilattice;
- (2) the semigroup  $v(X)$  is commutative;
- (3) the semigroup  $N_2(X)$  is commutative;
- (4) the semigroup  $\lambda(X)$  is commutative and  $(1, 2)$ -Clifford.

*Proof.* The implication (1)  $\Rightarrow$  (2) follows from Theorem 2.5 as  $\mathcal{A} * \mathcal{B} = \mathcal{A} \otimes \mathcal{B} = \mathcal{B} \otimes \mathcal{A} = \mathcal{B} * \mathcal{A}$  for every  $\mathcal{A}, \mathcal{B} \in v^\bullet(X) = v(X)$ .

The implication (2)  $\Rightarrow$  (3) is trivial.

(3)  $\Rightarrow$  (1) Assume that the semigroup  $N_2(X)$  is commutative. Then so is the semigroup  $X$ . Being a commutative band, the semigroup  $X$  is a semilattice. Assuming that  $X$  is not linear, we can find two points  $x, y \in X$  with  $xy \notin \{x, y\}$ . It can be shown that the linked upfamilies  $\mathcal{A} = \langle \{x, y\} \rangle$  and  $\mathcal{B} = \langle \{x, xy\}, \{y, xy\} \rangle \in N_k(X)$  do not commute because  $\{xy\} \in \mathcal{A} * \mathcal{B} \setminus \mathcal{B} * \mathcal{A}$ . Therefore,  $X$  is a linear semilattice. Since  $\beta(X) \subset v(X)$  is commutative, Corollary 2.2 implies that the linear semilattice  $X$  is finite.

(1)  $\Leftrightarrow$  (4) If  $X$  is a finite linear semilattice, then  $\lambda(X)$  is commutative by the implication (1)  $\Rightarrow$  (2) of this theorem and is  $(1, 2)$ -Clifford by Theorem 1.3.

If the semigroup  $\lambda(X)$  is commutative and  $(1, 2)$ -Clifford, then the semigroup  $X \subset \lambda(X)$  is commutative and by Theorem 1.3,  $X$  is linear. By Corollary 2.2, the linear semilattice  $X$  is finite. □

Now we shall characterize semilattices  $X$  with commutative superextension  $\lambda(X)$ . A semilattice  $X$  is called a *bush* if for any maximal linear subsemilattices  $A, B \subset X$  the product  $A * B$  is the singleton  $\{\min X\}$  containing the smallest element  $\min X$  of  $X$ . This definition implies that  $A \cap B = A * B = \{\min X\}$ . By a *branch* of a bush  $X$  we understand a maximal linear subsemilattice of  $X$ .

**Theorem 2.7.** *A semilattice  $X$  has commutative superextension  $\lambda(X)$  if and only if  $X$  is a bush with finite branches.*

*Proof.* First assume that  $X$  is a bush with finite branches, and take any two maximal linked systems  $\mathcal{A}, \mathcal{B} \in \lambda(X)$ . Since the products  $\mathcal{A} * \mathcal{B}$  and  $\mathcal{B} * \mathcal{A}$  are maximal linked upfamilies, the equality  $\mathcal{A} * \mathcal{B} = \mathcal{B} * \mathcal{A}$  will follow as soon as we check that any two basic sets  $C_{AB} = \bigcup_{a \in A} a * B_a \in \mathcal{A} * \mathcal{B}$  and  $C_{BA} = \bigcup_{b \in B} b * A_b \in \mathcal{B} * \mathcal{A}$  have non-empty

intersection. Here  $A \in \mathcal{A}$ ,  $(B_a)_{a \in A} \in \mathcal{B}^A$ ,  $B \in \mathcal{B}$ , and  $(A_b)_{b \in B} \in \mathcal{A}^B$ . Assume conversely that  $C_{AB} \cap C_{BA} = \emptyset$ . Then either  $\min X \notin C_{AB}$  or  $\min X \notin C_{BA}$ .

Without loss of generality,  $\min X \notin C_{AB}$ . Then  $\min X \notin A$  and for each  $a \in A$  the set  $\{a\} \cup B_a$  lies in a branch of  $X$ . Since branches of  $X$  meet only at the point  $\min X$ , all the sets  $\{a\} \cup B_a$ ,  $a \in A$ , lie in the same (finite) branch. Repeating the argument of Theorem 2.5, we can show that  $C_{AB} \supset AB'$  for some set  $B' \in \mathcal{B}$ . Since  $\mathcal{B}$  is linked, there is a point  $b \in B \cap B'$ . By the same reason, there is a point  $a \in A \cap A_b$ . Then  $ab = ba \in AB' \cap bA_b \subset C_{AB} \cap C_{BA}$  and we are done.

Now assume that  $X$  is a semilattice with commutative superextension  $\lambda(X)$ . Corollary 2.2 implies that all branches of  $X$  are finite. We claim that for every  $z \in X$  the lower set  $\downarrow z = \{x \in X : xz = x\}$  is linear. Assuming the converse, find two points  $x, y \in \downarrow z$  such that  $xy \notin \{x, y\}$ . It follows that the points  $x, y, z, xy$  are pairwise distinct. It is easy to check that the maximal linked upfamilies  $\mathcal{A} = \langle \{x, y\}, \{x, z\}, \{y, z\} \rangle$  and  $\mathcal{B} = \langle \{x, y\}, \{x, xy\}, \{y, xy\} \rangle$  do not commute because  $\{x, y\} \in \mathcal{B} * \mathcal{A} \setminus \mathcal{A} * \mathcal{B}$ . Thus  $\downarrow z$  is linear for every  $z \in X$ , which means that  $X$  is a tree.

Assuming that the tree  $X$  is not a bush, we can find two points  $x, y \in X$  such that  $xy \notin \{x, y, z\}$  where  $z = \min X$ . Now consider the maximal linked systems  $\mathcal{A} = \langle \{x, y\}, \{x, z\}, \{y, z\} \rangle$  and  $\mathcal{B} = \langle \{x, y\}, \{x, xy\}, \{y, xy\} \rangle$  and observe that they do not commute as  $\{xy\} \in \mathcal{A} * \mathcal{B}$  misses the set  $\{x, y, z\} \in \mathcal{B} * \mathcal{A}$ .  $\square$

### 3. SEMIGROUPS WHOSE EXTENSIONS ARE SEMILATTICES

In this section we shall characterize semigroups  $X$  whose extensions  $v(X)$ ,  $\lambda(X)$ ,  $\varphi(X)$ , or  $N_2(X)$  are semilattices.

**Theorem 3.1.** *For a semigroup  $X$  the following conditions are equivalent:*

- (1)  $X$  is finite linear semilattice;
- (2)  $v(X)$  is a semilattice;
- (3)  $\lambda(X)$  is a semilattice;
- (4)  $\varphi(X)$  is a semilattice.

*Proof.* (1)  $\Rightarrow$  (2) If  $X$  is a finite linear semilattice, then  $v(X)$  is a semilattice (=commutative band) by Theorems 1.1 and 2.6.

The implications (2)  $\Rightarrow$  (3, 4) are trivial.

The implication (3)  $\Rightarrow$  (1) follows from Theorems 1.1 and 2.7.

(4)  $\Rightarrow$  (1) Assume that  $\varphi(X)$  is a semilattice. Then  $X$ , being a subsemigroup of the commutative semigroup  $\varphi(X)$  is commutative. Since  $\varphi(X)$  is a band,  $X$  is a linear semigroup by Theorem 1.1. Thus  $X$ , being a commutative linear semigroup, is a linear semilattice. Since the subsemigroup  $\beta(X) \subset \lambda(X)$  is commutative, the linear semilattice  $X$  is finite by Corollary 2.2.  $\square$

### 4. SEMIGROUPS WHOSE EXTENSIONS ARE LINEAR

In this section we characterize semigroups  $X$  whose extensions  $v(X)$ ,  $\lambda(X)$  or  $\varphi(X)$  are linear semigroups.

A semigroup  $S$  is called a *semigroup of left (right) zeros* if  $xy = x$  (resp.  $xy = y$ ) for all  $x, y \in X$ .

**Theorem 4.1.** *For a semigroup  $X$  the semigroup  $v(X)$  is linear if and only if  $X$  is either a semigroup of right zeros or a semigroup of left zeros.*

*Proof.* If  $X$  is a semigroup of left zeros, then for any upfamilies  $\mathcal{A}, \mathcal{B} \in v(X)$  and any basic element  $\bigcup_{x \in A} xB_x \in \mathcal{A} * \mathcal{B}$  we get  $\bigcup_{x \in A} xB_x = \bigcup_{x \in A} \{x\} = A$  and thus  $\mathcal{A} * \mathcal{B} \subset \mathcal{A}$ . On the other hand, each  $A \in \mathcal{A}$  belongs to  $\mathcal{A} * \mathcal{B}$  as  $A = A * B \in \mathcal{A} * \mathcal{B}$  for any  $B \in \mathcal{B}$ .

Assume that the semigroup  $v(X)$  is linear. Then  $X$ , being a subsemigroup of  $v(X)$ , also is linear. Let  $x, y$  be any two distinct elements of  $X$ . First we prove that  $xy \neq yx$ . Assume conversely that  $xy = yx$ . Then  $xy = yx \in \{x, y\}$  and we lose no generality assuming that  $xy = x$ . Now consider two upfamilies  $\mathcal{A} = \langle \{x, y\} \rangle$  and  $\mathcal{B} = \langle \{x\}, \{y\} \rangle$  and observe that

$$\mathcal{B} * \mathcal{A} = \langle \{xx, xy\}, \{yx, yy\} \rangle = \langle \{x\}, \{x, y\} \rangle = \langle \{x\} \rangle \notin \{\mathcal{A}, \mathcal{B}\},$$

so  $v(X)$  is not linear and this is a required contradiction.

Thus  $xy \neq yx$  for all distinct points  $x, y \in X$ . We call a pair  $(x, y) \in X^2$  *left* if  $xy = x$  and  $yx = y$  and *right* if  $xy = y$  and  $yx = x$ . Since  $X$  is linear, each pair  $(x, y) \in X^2$  is either left or right. We claim that either all pairs  $(x, y) \in X^2$  are left or else all such pairs are right. Assuming the opposite, find pairs  $(x, y), (a, b) \in X^2$  such that  $(x, y)$  is not left and  $(a, b)$  is not right. Then  $x \neq y, a \neq b$  and the pair  $(x, y)$  is right while  $(a, b)$  is left. Consider the filters  $\mathcal{A} = \langle \{x, a\} \rangle$  and  $\mathcal{B} = \langle \{y, b\} \rangle$  and observe that  $\mathcal{A} * \mathcal{B} = \langle \{xy, xb, ay, ab\} \rangle = \langle \{y, xb, ay, a\} \rangle$ . Since  $v(X)$  is linear, either  $\mathcal{A} * \mathcal{B} = \mathcal{A}$  or  $\mathcal{A} * \mathcal{B} = \mathcal{B}$ . In the first case  $\{x, a\} \supset \{y, xb, ay, a\} \supset \{y, a\}$  and hence  $y = a$ . In the second case,  $\{y, a\} \subset \{y, b\}$  and thus  $a = y$ . Now consider the filters  $\mathcal{C} = \langle \{x, b\} \rangle$  and  $\mathcal{D} = \langle \{a\} \rangle$  and observe that  $\mathcal{C} * \mathcal{D} = \langle \{xa, ba\} \rangle = \langle \{xy, b\} \rangle = \langle \{y, b\} \rangle = \langle \{a, b\} \rangle \notin \{\mathcal{C}, \mathcal{D}\}$ , which contradicts the linearity of  $v(X)$ .

Therefore either each pair  $(x, y) \in X^2$  is left and then  $X$  is a semigroup of left zeros or else each pair  $(x, y) \in X^2$  is right and then  $X$  is a semigroup of right zeros.  $\square$

**Theorem 4.2.** *For a semigroup  $X$  the following conditions are equivalent:*

- (1) *the semigroup  $\varphi(X)$  is linear;*
- (2) *the semigroup  $N_2(X)$  is linear;*
- (3) *either  $X$  is a semigroup of left zeros or  $X$  is a semigroup of right zeros or else  $X$  is a semilattice of order  $|X| \leq 2$ .*

*Proof.* (3)  $\Rightarrow$  (2) If  $|X| = 1$ , then  $N_2(X)$  is a singleton and hence is a linear semigroup. If  $X$  is a semilattice of order  $|X| = 2$ , then  $X = \{0, 1\}$  for some elements  $0, 1$  with  $0 \cdot 1 = 1 \cdot 0 = 0$ . In this case  $N_2(X) = \varphi(X)$  is a 3-element linear semilattice ordered as:

$$\langle \{0\} \rangle \leq \langle \{0, 1\} \rangle \leq \langle \{1\} \rangle.$$

If  $X$  is a semigroup of left or right zeros, then the semigroup  $v(X)$  is linear by Theorem 4.1 and so is its subsemigroup  $N_2(X)$ .

(2)  $\Rightarrow$  (1) Is the semigroup  $N_2(X)$  is linear, then so is its subsemigroup  $\varphi(X)$ .

(1)  $\Rightarrow$  (3) Assume that the semigroup  $\varphi(X)$  is linear. Then  $X$ , being a subsemigroup of  $\varphi(X)$ , is linear as well. If  $|X| \leq 2$ , then either  $X$  is a linear semilattice or a semigroup of left or right zeros. So, we assume that  $|X| \geq 3$ . We claim that distinct elements  $x, y \in X$  do not commute. Assume conversely that  $xy = yx$  for some distinct elements  $x, y \in X$ . Since  $xy = yx \in \{x, y\}$  we lose no generality assuming that  $xy = yx = x$ . Fix any element  $z \in X \setminus \{x, y\}$ . Now consider 3 cases:

1.  $zx = z$ . In this case we can consider the filters  $\mathcal{A} = \langle \{z, y\} \rangle$  and  $\mathcal{B} = \langle \{x, y\} \rangle$  and observe that  $\mathcal{A} * \mathcal{B} = \langle \{zx, yx, zy, yy\} \rangle = \langle \{z, x, zy, y\} \rangle \notin \{\mathcal{A}, \mathcal{B}\}$ , which contradicts the linearity of  $\varphi(X)$ .
2.  $zx = x$  and  $zy = z$ . In this case we can consider the filters  $\mathcal{A} = \langle \{z, y\} \rangle$  and  $\mathcal{B} = \langle \{x, y\} \rangle$  and observe that  $\mathcal{A} * \mathcal{B} = \langle \{zx, yx, zy, yy\} \rangle = \langle \{x, x, z, y\} \rangle \notin \{\mathcal{A}, \mathcal{B}\}$ , which contradicts the linearity of  $\varphi(X)$ .
3.  $zx = x$  and  $zy = y$ . In this case we can consider the filters  $\mathcal{A} = \langle \{x, z\} \rangle$  and  $\mathcal{B} = \langle \{y, z\} \rangle$  and observe that  $\mathcal{A} * \mathcal{B} = \langle \{xy, xz, zy, zz\} \rangle = \langle \{x, xz, y, z\} \rangle \notin \{\mathcal{A}, \mathcal{B}\}$ , which again contradicts the linearity of  $\varphi(X)$ .

Those contradictions show that distinct elements of  $X$  do not commute. Continuing as in the proof of Theorem 4.1, we can show that  $X$  is a semigroup of right or left zeros.  $\square$

Finally, we characterize commutative semigroups with linear superextensions.



**Theorem 4.3.** *For a commutative semigroup  $X$  the semigroup  $\lambda(X)$  is linear if and only if  $X$  is a linear semilattice of order  $|X| \leq 3$ .*

*Proof.* If  $X$  is a linear semilattice of order  $|X| \leq 2$ , then the semigroup  $\lambda(X) = X$  is linear.

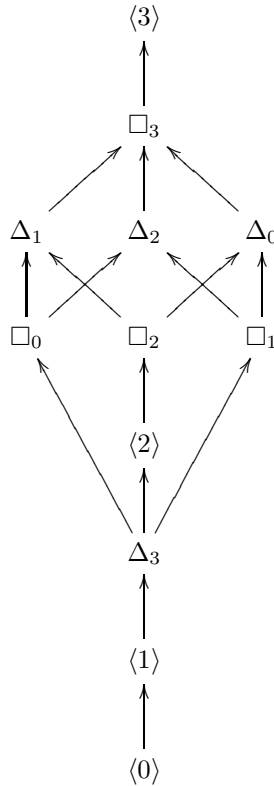
If  $X$  is a linear semilattice of order  $|X| = 3$ , then  $X$  can be identified with the set  $3 = \{0, 1, 2\}$  endowed with the operation  $xy = \min\{x, y\}$ . The semigroup  $\lambda(X)$  contains 4 elements:  $0, 1, 2$  and  $\Delta = \{A \subset 3 : |A| \geq 2\}$ . One can check that  $\lambda(3)$  is a linear semilattice ordered as follows:

$$0 \leq \Delta \leq 1 \leq 2.$$

This proves the “if” part of the theorem. To prove the “only if” part we first shall analyze the structure of the superextension  $\lambda(4)$  of the semilattice  $4 = \{0, 1, 2, 3\}$  endowed with the operation  $xy = \min\{x, y\}$ . By Theorem 3.1,  $\lambda(4)$  is a semilattice. It contains 12 elements:

$$\langle k \rangle, \Delta_k = \langle \{A \subset n : |A| = 2, k \notin A\} \rangle \text{ and } \square_k = \langle \{n \setminus \{k\}, A : A \subset n, |A| = 2, k \in A\} \rangle \text{ where } k \in 4.$$

The order structure of the semilattice  $\lambda(4)$  is described in the following diagram:



Looking at this diagram we see that the semilattice  $\lambda(4)$  is not linear.

Now assume that  $X$  is a commutative semigroup whose superextension  $\lambda(X)$  is linear. Then  $X$  is a linear semilattice. If  $|X| > 3$ , then  $\lambda(X)$  is not linear as it contains a subsemigroup isomorphic to the semilattice  $\lambda(4)$ , which is not linear.  $\square$

## 5. LATTICES WHOSE EXTENSIONS ARE LATTICES

In this section we characterize lattices whose extensions  $v(X)$ ,  $\lambda(X)$  or  $\varphi(X)$  are lattices.

A *lattice* is a set  $X$  endowed with two semilattice operations  $\wedge, \vee : X \times X \rightarrow X$  such that  $(x \wedge y) \vee y = y$  and  $(x \vee y) \wedge y = y$  for all  $x, y \in X$ .

Both operations  $\wedge$  and  $\vee$  of a lattice  $X$  can be extended to right-topological operations  $\wedge$  and  $\vee$  on the compact Hausdorff space  $v(X)$ . Is it natural to ask if the triple  $(v(X), \wedge, \vee)$  is a lattice.

A lattice will be called *linear* if  $x \wedge y, x \vee y \in \{x, y\}$  for all  $x, y \in X$ .

**Theorem 5.1.** *For a lattice  $X$  the following conditions are equivalent:*

- (1)  $X$  is a linear lattice of order  $|X| \leq 2$ .
- (2)  $v(X)$  is a lattice;
- (3)  $\lambda(X)$  is a lattice;
- (4)  $\varphi(X)$  is a lattice.

*Proof.* (1)  $\Rightarrow$  (2) If  $X$  is a linear lattice of order  $|X| = 1$ , then  $v(X) = X$  is a trivial lattice. If  $X$  is a linear lattice of order 2, then  $X$  can be identified with the lattice  $2 = \{0, 1\}$  endowed with the operations  $x \wedge y = \min\{x, y\}$  and  $x \vee y = \max\{x, y\}$ . In this case  $\lambda(2) = \beta(2)$  coincides with the lattice 2,  $\varphi(2) = \{\langle\{0\}\rangle, \langle\{0, 1\}\rangle, \langle\{1\}\rangle\}$  is a 3-element lattice, isomorphic to the lattice  $3 = \{0, 1, 2\}$  endowed with the operations  $\min$  and  $\max$ , and  $v(2) = \{\langle\{0\}\rangle, \langle\{0, 1\}\rangle, \langle\{0\}, \{1\}\rangle, \langle\{1\}\rangle\}$  is a 4-element lattice isomorphic to the lattice  $\{0, 1\}^2$ .

The implications (2)  $\Rightarrow$  (3, 4) are trivial.

(3, 4)  $\Rightarrow$  (1) Assume that  $\lambda(X)$  or  $\varphi(X)$  is a lattice. By Theorem 3.1, the lattice  $X$  is finite and linear. We claim that  $|X| \leq 2$ . Assuming the converse, we conclude that the lattice  $X$  contains a sublattice isomorphic to the lattice  $(3, \min, \max)$ .

Consider the maximal linked upfamily  $\Delta = \{A \subset 3 : |A| \geq 2\}$  and observe that

$$\max\{\Delta, \langle 1 \rangle\} = \langle 1 \rangle = \min\{\Delta, \langle 1 \rangle\},$$

which implies that  $\lambda(3)$  is not a lattice and then  $\lambda(X)$  also is not a lattice.

Next, consider the filters  $\mathcal{A} = \langle\{0, 1, 2\}\rangle$  and  $\mathcal{B} = \langle\{0, 2\}\rangle$  and observe that

$$\max\{\mathcal{A}, \mathcal{B}\} = \mathcal{A} = \min\{\mathcal{A}, \mathcal{B}\}$$

implying that  $\varphi(3)$  is not a lattice and then  $\varphi(X)$  also cannot be a lattice. □

## REFERENCES

- [1] T. Banakh, V. Gavrylkiv, O. Nykyforchyn, *Algebra in superextensions of groups, I: zeros and commutativity*, Algebra Discrete Math. (2008), No.3, 1–29.
- [2] T. Banakh, V. Gavrylkiv. *Algebra in superextension of groups, II: cancelativity and centers*, Algebra Discrete Math. (2008), No.4, 1–14.
- [3] T. Banakh, V. Gavrylkiv. *Algebra in superextension of groups: the minimal ideal of  $\lambda(G)$* , Mat. Stud. **31** (2009), 142–148.
- [4] T. Banakh, V. Gavrylkiv. *Algebra in the superextensions of twinic groups*, Dissert. Math. **473** (2010), 74pp.
- [5] T. Banakh, V. Gavrylkiv. *The superextensions of inverse semigroups*, preprint.
- [6] A.H. Clifford, G.B. Preston, The algebraic theory of semigroups. Vol. I., Mathematical Surveys. **7**. AMS, Providence, RI, 1961.
- [7] V. Gavrylkiv. *The spaces of inclusion hyperspaces over noncompact spaces*, Matem. Studii. **28:1** (2007), 92–110.
- [8] V. Gavrylkiv, *Right-topological semigroup operations on inclusion hyperspaces*, Mat. Stud. **29:1** (2008), 18–34.
- [9] N. Hindman, D. Strauss, Algebra in the Stone-Čech compactification, de Gruyter, Berlin, New York, 1998.
- [10] J. van Mill, Supercompactness and Wallman spaces, Math. Centre Tracts. **85**. Amsterdam: Math. Centrum., 1977.
- [11] I. Protasov, Combinatorics of Numbers, VNTL, Lviv, 1997.
- [12] C. Schubert, G. Seal, *Extensions in the theory of Lax algebra*, Theory and Appl. of Categories, **21:7** (2008), 118–151.
- [13] A. Teleiko, M. Zarichnyi. Categorical Topology of Compact Hausdoff Spaces, VNTL, Lviv, 1999.
- [14] A. Verbeek. Superextensions of topological spaces. MC Tract 41, Amsterdam, 1972.

IVAN FRANKO NATIONAL UNIVERSITY OF LVIV, UKRAINE AND  
 UNIWERSYTET HUMANISTYCZNO-PRZYRODNICZY JANA KOCHANOWSKIEGO, KIELCE, POLAND  
*E-mail address:* t.o.banakh@gmail.com

VASYL STEFANYK PRECARPATHIAN NATIONAL UNIVERSITY, IVANO-FRANKIVSK, UKRAINE  
*E-mail address:* vgavrylkiv@yahoo.com